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1992 J. Phys. A: Math. Gen. 25 L419

(http://iopscience.iop.org/0305-4470/25/8/007)

J. Phys. A: Math. Gen. 25 (1992) L419–L426. Printed in the UK

LETTER TO THE EDITOR

On some solutions of the two-dimensional sine-Gordon equation

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Received 6 November 1991

Abstract. There exists an approach for finding of exact solutions of the two-dimensional sine-Gordon equation. Using this approach three classes of solutions have been found. One of the classes consists of running wave solutions which are a generalization of the solutions of the one-dimensional sine-Gordon equation. This class of solutions is studied here.

The one-dimensional sine-Gordon equation

$$\phi_{xx} - \phi_{tt} = \sin\phi(x, t) \tag{1}$$

is well known because it possesses soliton solutions [1] and has many physical and mathematical applications. This equation has arisen in differential geometry [2]. Now the one-dimensional sine-Gordon equation is a pattern equation for description of self-induced transparency [3], for description of magnetic domain wall dynamics [4], and so on [5]. A correspondence between the standing wave solutions of this equation and the self-consistent two-dimensional Poisson–Boltzmann structures also exists [6].

An approach for solving the two-dimensional sine-Gordon equation:

$$\phi_{xx} + \phi_{yy} - \phi_{tt} = \sin\phi(x, y, t) \tag{2}$$

was found. The solutions found are of the type

$$\phi(x, y, t) = 4\tan^{-1}[Af(\alpha x; k_1)g(\beta y + \delta\gamma t; k_2)] \tag{3}$$

where $\delta = 1$ or $\delta = -1$, f and g are real Jacobi elliptic functions and k_1 and k_2 are their corresponding elliptic integral modules; α, β and γ are parameters.

The generating equations of the Jacobi elliptic functions are:

$$\begin{aligned}(f')^2 &= a_1 f^4 + b_1 f^2 + c_1\\ (g')^2 &= a_2 g^4 + b_2 g^2 + c_2\end{aligned} \tag{4}$$

where a_1, b_1, c_1, a_2, b_2, c_2, are parameters. If (4) is substituted into (2) the result is:

$$\begin{aligned}2f^3g[\alpha^3 a_1 - (\beta^2 - \delta^2\gamma^2)A^2c_2] + fg[\alpha^2 b_1 + (\beta^2 - \delta^2\gamma^2)b_2 - 1]\\ + 2fg^3[(\beta^2 - \delta^2\gamma^2)a_2 - \alpha^2A^2c_1]\\ + f^3g^3[A^2 - \alpha^2A^2b_1 - (\beta^2 - \delta^2\gamma^2)A^2b_2] = 0.\end{aligned} \tag{5}$$

If

$$\alpha^2 b_1 - (\delta^2\gamma - \beta^2)b_2 = 1 \tag{6a}$$

$$\alpha^2 a_1 + (\delta^2\gamma^2 - \beta^2)A^2c_2 = 0 \tag{6b}$$

$$(\delta^2\gamma^2 - \beta^2)a_2 + \alpha^2A^2c_1 = 0 \tag{6c}$$

0305-4470/92/080419+08$04.50

then (3) is a solution of the two-dimensional sine-Gordon equation. The relations ($6a$–c) are characteristic relations for the solution (3). Due to these relations three classes of solutions of the equation (2) were found. One of the classes consists of running wave solutions which are a generalization of the solutions of the one-dimensional sine-Gordon equation. For this class of solutions

$$\delta^2\gamma^2-\beta^2>0. \tag{7}$$

Let $\bar{\gamma}^2=\delta^2\gamma^2-\beta^2$. Then the form of the system of relations (6) is as follows:

$$\alpha^2 b_1-\bar{\gamma}^2 b_2=1 \tag{8a}$$

$$\alpha^2 a_1+\bar{\gamma}^2A^2c_2=0 \tag{8b}$$

$$\bar{\gamma}^2 a_2+\alpha^2A^2c_1=0. \tag{8c}$$

The system ($8a$–c) is similar to the system of characteristic relations for the one-dimensional sine-Gordon equation. If $\beta=0$ then the system (8) and the system of characteristic relations for the one-dimensional sine-Gordon equation are the same. In this case the solution (3) of the two-dimensional sine-Gordon equation is reduced to the solution

$$\bar{\phi}(x,t)=4\tan^{-1}[Af(\alpha x;k_1)g(\delta\gamma t;k_2)] \tag{9}$$

of the one-dimensional sine-Gordon equation.

The solution (3) of the two-dimensional sine-Gordon equation depends on six parameters: α, β, γ, A, k_1, k_2, among which the three relations ($6a$–c) exist i.e. the solution depends on three free parameters. The solution (9) of the one-dimensional sine-Gordon equation depends on two free parameters. In this case the third parameter β is fixed: $\beta=0$.

The parameters a_1, b_1, c_1 depend on the elliptic integral module k_1 and the parameters a_2, b_2, c_2 depend on the module k_2.

Due to the relations ($6a$–c) and (7) the following seven solutions of the two-dimensional sine-Gordon equation were found:

$$\phi_1=4\tan^{-1}[A\,\mathrm{cn}(\alpha x;k_1)\,\mathrm{cn}(\beta y+\delta\gamma t;k_2)$$

$$k_1^2=\frac{A^2[\alpha^2(A^2+1)+1]}{\alpha^2(A^2+1)^2}\qquad k_2^2=\frac{A^2[(\delta^2\gamma^2-\beta^2)(A^2+1)-1]}{(\delta^2\gamma^2-\beta^2)(A^2+1)^2} \tag{10}$$

$$\alpha^2-(\delta^2\gamma^2-\beta^2)=\frac{A^2-1}{A^2+1}$$

$$\phi_2=4\tan^{-1}[A\,\mathrm{dn}(\alpha x;k_1)\,\mathrm{sn}(\beta y+\delta\gamma t;k_2)]$$

$$k_2^2=1-\frac{1-\alpha^2(A^2+1)/A^2}{\alpha^2(A^2+1)}\qquad k_2^2=\frac{A^2[1-(\delta^2\gamma^2-\beta^2)(a^2+1)]}{(\delta^2\gamma^2-\beta^2)(A^2+1)} \tag{11}$$

$$\alpha^2=A^2(\delta^2\gamma-\beta^2)$$

$$\phi_3=4\tan^{-1}\left[A\,\mathrm{dn}(\alpha x;k_1)\frac{\mathrm{sn}(\beta y+\delta\gamma t;k_2)}{\mathrm{cn}(\beta y+\delta\gamma t;k_2)}\right]$$

$$k_1^2=1-\frac{\alpha^2(A^2-1)/A^2-1}{\alpha^2(a^2-1)^2}\qquad k_2^2=1-\frac{A^2[(\delta^2\gamma^2-\beta^2)(A^2-1)-1]}{(\delta^2\gamma^2-\beta^2)(A^2-1)} \tag{12}$$

$$\alpha^2=A^2(\delta^2\gamma^2-\beta^2)$$

$$\phi_4 = 4\tan^{-1}\left[A\,\mathrm{dn}(\alpha x;\,k_1)\frac{1}{\mathrm{sn}(\beta y+\delta\gamma t;\,k_2)}\right]$$

$$k_1^2 = \frac{\alpha^2(A^2+1)^2 - A^2}{\alpha^2 A^2(A^2+1)} \qquad k_2^2 = 1 - \frac{A^2-(\delta^2\gamma^2-\beta^2)(A^2+1)}{A^2(\delta^2\gamma^2-\beta^2)(A^2+1)} \tag{13}$$

$$\alpha^2 + (\delta^2\gamma^2-\beta^2) = \frac{A^2}{A^2+1}$$

$$\phi_5 = 4\tan^{-1}\left[A\,\mathrm{dn}(\alpha x;\,k_1)\frac{\mathrm{cn}(\beta y+\delta\gamma t;\,k_2)}{\mathrm{sn}(\beta y+\delta\gamma t;\,k_2)}\right]$$

$$k_1^2 = 1 - \frac{1+\alpha^2(1-A^2)/A^2}{\alpha^2(1-A^2)} \qquad k_2^2 = 1 + \frac{1-(\delta^2\gamma^2-\beta^2)(1-A^2)/A^2}{(\delta^2\gamma^2-\beta^2)(1-A^2)} \tag{14}$$

$$\alpha^2 - (\delta^2\gamma^2-\beta^2) = \frac{A^2}{A^2-1}$$

$$\phi_6 = 4\tan^{-1}\left[A\frac{\mathrm{sn}(\alpha x;\,k_1)}{\mathrm{cn}(\alpha x;\,k_1)}\mathrm{dn}(\beta y+\delta\gamma t;\,k_2)\right]$$

$$k_1^2 = \frac{\alpha^2(1-A^2)^2 + A^2}{\alpha^2(1-A^2)} \qquad k_2^2 = \frac{A^2-(\delta^2\gamma 2-\beta^2)(1-A^2)^2}{(\delta^2\gamma^2-\beta^2)A^2(1-A^2)} \tag{15}$$

$$\delta^2\gamma^2 - \beta^2 = A^2\alpha^2$$

$$\phi_7 = 4\tan^{-1}\left[A\frac{\mathrm{sn}(\alpha x;\,k_1)}{\mathrm{cn}(\alpha x;\,k_1)}\frac{1}{\mathrm{dn}(\beta y+\delta\gamma t;\,k_2)}\right]$$

$$k_1^2 = \frac{\alpha^2(1-A^2)^2 + A^2}{\alpha^2(1-A2)} \qquad k_2^2 = \frac{(\delta^2\gamma^2-\beta^2)(1-A^2)^2 - A^2}{(\delta^2\gamma^2-\beta^2)(1-A^2)} \tag{16}$$

$$\alpha^2 - (\delta^2\gamma^2-\beta^2) = \frac{1}{1-A^2}.$$

These seven solutions are distributed among the following four classes:

Class A: the solution ϕ_1

Class B: the solutions ϕ_2 and ϕ_4

Class C: the solutions ϕ_3 and ϕ_5

Class D: the solutions ϕ_6 and ϕ_7.

These classes of solutions are analogous to the corresponding classes of solutions of the one-dimensional sine-Gordon equation.

Class A. Let $\beta = 0$. Then the solution ϕ_1 is reduced to the corresponding solution of the one-dimensional sine-Gordon equation

$$\phi_1 = \bar{\phi}_1 = 4\tan^{-1}[A\,\mathrm{cn}(\alpha x;\,k_1)\,\mathrm{cn}(\delta\gamma t;\,k_2)]$$

$$k_1^2 = \frac{A^2[\alpha^2(A^2+1)+1]}{\alpha^2(A^2+1)^2} \qquad k_2^2 = \frac{A^2[\delta^2\gamma^2(A^2+1)-1]}{\delta^2\gamma^2(A^2+1)^2} \tag{17}$$

$$\alpha^2 - \delta^2\gamma^2 = \frac{A^2-1}{A^2+1}.$$

The solution $\bar{\phi}_1$ describes the plasma oscillations in one-dimensional Josephson junction. The solution ϕ_1 of the two-dimensional sine-Gordon equation is a generalization about the solution $\bar{\phi}_1$

Class B. The solutions ϕ_2 and ϕ_4 are generalizations of the solutions belong to the class of solutions of the one-dimensional sine-Gordon equation describing the breather oscillations in the Josephson junction. If $\beta=0$ the solution ϕ_2 is reduced to:

$$\phi_2=\bar{\phi}_2=4\tan^{-1}[A\,\mathrm{dn}(\alpha x;k_1)\,\mathrm{sn}(\delta\gamma t;k_2)]$$

$$k_1^2=\frac{\alpha^2(A^2+1)-A^2}{\alpha^2A^2(A^2+1)}\qquad k_2^2=\frac{A^2[1-\delta^2\gamma^2(A^2+1)]}{\delta^2\gamma^2(A^2+1)}\qquad \alpha^2=A^2\delta^2\gamma^2. \tag{18}$$

The solution ϕ_4 is reduced to the solution:

$$\phi_4=\bar{\phi}_4=4\tan^{-1}\left[A\,\mathrm{dn}(\alpha x;k_1)\frac{1}{\mathrm{sn}(\delta\gamma t;k_2)}\right]$$

$$k_2^2=\frac{\alpha^2(A^2+1)-A^2}{\alpha^2A^2(A^2+1)}\qquad k_2^2=\frac{A^2-\delta^2\gamma^2(A^2+1)}{A^2\delta^2\gamma(A^2+1)}\qquad \alpha^2+\delta^2\gamma^2=\frac{A^2}{A^2+1}. \tag{19}$$

Class C. If $\beta=0$ then the solutions ϕ_3 and ϕ_5 are reduced to the following solutions of the one-dimensional sine-Gordon equation

$$\phi_3=\bar{\phi}_3=4\tan^{-1}\left[A\,\mathrm{dn}(\alpha x;k_1)\frac{\mathrm{sn}(\delta\gamma t;k_2)}{\mathrm{cn}(\delta\gamma t;k_2)}\right]$$

$$k_1^2=1-\frac{\alpha^2(A^2-1)/A^2-1}{\alpha^2(A^2-1)^2}\qquad k_2^2=1-\frac{A^2[\delta^2\gamma^2(A^2-1)-1]}{\delta^2\gamma^2(A^2-1)}\qquad \alpha^2=A^2\delta^2\gamma^2 \tag{20}$$

$$\phi_5=\bar{\phi}_5=4\tan^{-1}\left[A\,\mathrm{dn}(\alpha x;k_1)\frac{\mathrm{cn}(\delta\gamma t;k_2)}{\mathrm{sn}(\delta\gamma t;k_2)}\right]$$

$$k_1^2=1-\frac{1+\alpha^2(1-A^2)/A^2}{\alpha^2(1-A^2)}\qquad k_2^2=1+\frac{1-\delta^2\gamma^2(1-A^2)/A^2}{\delta^2\gamma^2(1-A^2)} \tag{21}$$

$$\alpha^2-\delta^2\gamma^2=\frac{A^2}{A^2-1}.$$

The solutions $\bar{\phi}_3$ and $\bar{\phi}_5$ describe the fluxon oscillations in one-dimensional Josephson junction and the solutions ϕ_3 and ϕ_5 are their two-dimensional generlizations.
Class D. The solutions ϕ_6 and ϕ_7 belong to this class. If $\beta=0$ these solutions are reduced to the following solutions of the one-dimensional sine-Gordon equation:

$$\bar{\phi}_6=4\tan^{-1}\left[A\frac{\mathrm{sn}(\alpha x;k_1)}{\mathrm{cn}(\alpha x;k_1)}\mathrm{dn}(\delta\gamma t;k_2)\right]$$

$$k_1^2=\frac{\alpha^2(1-A^2)^2+A^2}{\alpha^2(1-A^2)}\qquad k_2^2=\frac{A^2-\delta^2\gamma^2(1-A^2)^2}{\delta^2\gamma^2A^2(1-A^2)}\qquad \delta^2\gamma^2=A^2\alpha^2 \tag{22}$$

$$\phi_7=4\tan^{-1}\left[A\frac{\mathrm{sn}(\alpha x;k_1)}{\mathrm{cn}(\alpha x;k_1)}\frac{1}{\mathrm{dn}(\delta\gamma t;k_2)}\right]$$

$$k_1^2=\frac{\alpha^2(1-A^2)^2+A^2}{\alpha^2(1-A^2)}\qquad k_2^2=\frac{\delta^2\gamma^2(1-A^2)^2-A^2}{\delta^2\gamma^2(1-A^2)} \tag{23}$$

$$\alpha^2-\delta^2\gamma^2=\frac{1}{1-A^2}.$$

The solutions of the two-dimensional sine-Gordon equation (10)-(16) describe running waves. As opposed to this the solutions of the one-dimensional sine-Gordon equation describe standing waves. All the same the solutions of the two-dimensional sine-Gordon equation are a generalization of the solutions of the one-dimensional sine-Gordon equation.

Special cases for the solutions of the equation (2) exist. In these cases the modules of the Jacobi ellitic functions $k_1=0$ or $k_2=0$ or $k_1=1$ or $k_2=1$ or $k_1=0$, $k_2=0$ or $k_1=0$, $k_2=1$, or $k_1=1$, $k_2=0$ or $k_1=k_2=1$. (The modules are limited bilaterally: $0 \leqslant k_{1,2} \leqslant 1$). If the module $k=0$ the Jacobi elliptic functions are reduced to the following functions:

$$\mathrm{sn}(x;0)=\sin(x) \qquad \mathrm{cn}(x;0)=\cos(x) \qquad \mathrm{dn}(x;0)=1. \tag{25}$$

If the module $k=1$ the Jacobi elliptic functions are reduced to the functions:

$$\mathrm{sn}(x;1)=\tanh(x) \qquad \mathrm{cn}(x;1)=\mathrm{dn}(x;1)=1/\cosh(x). \tag{26}$$

Some of the possible special cases are given below.

Class A. Due to the system (6) the special case $k_1=k_2=0$ was found. Then

$$\alpha^2=\frac{A^2}{A^2+1} \qquad \text{and} \qquad \delta^2\gamma^2=\beta^2+\frac{1}{A^2+1}.$$

In this case the solution ϕ_1 is reduced to the following solution of the one-dimensional sine-Gordon equation:

$$\phi_1=\phi_{1,1}=4\tan^{-1}\left\{A\frac{\cos\left[\beta y\pm\left(\beta^2+\frac{1}{A^2+1}\right)^{1/2}t\right]}{\cosh\left[\left(\frac{A^2}{A^2+1}\right)^{1/2}x\right]}\right\}. \tag{27}$$

If $\beta=0$ it can be seen that $\phi_{1,1}$ is reduced to the following solution of the one-dimensional sine-Gordon equation:

$$\bar{\phi}_{1,1}=4\tan^{-1}\left\{A\frac{\cos\left[\pm\left(\frac{1}{A^2+1}\right)^{1/2}t\right]}{\cosh\left[\left(\frac{A^2}{A^2+1}\right)^{1/2}x\right]}\right\}. \tag{28}$$

The solution $\bar{\phi}_{1,1}$ is known as a breather, so that the solution $\phi_{1,1}$ is a generalization about the breather solution of equation (1). But the solution $\phi_{1,1}$ does not describe a standing wave; it describes a running wave.

Class B. The solution ϕ_2: two special cases are possible here:

1. $k_1=1$, $k_2=0$. Then

$$\alpha^2=\frac{A^2}{A^2+1} \qquad \delta^2\gamma^2=\beta^2+\frac{1}{A^2+1}$$

and the solution ϕ_2 is reduced to the following solution of the two-dimensional sine-Gordon equation:

$$\phi_1=\phi_{1,1}=4\tan^{-1}\left\{A\frac{\sin\left[\beta y\pm\left(\beta^2+\frac{1}{A^2+1}\right)^{1/2}t\right]}{\cosh\left[\left(\frac{A^2}{A^2+1}\right)^{1/2}x\right]}\right\}. \tag{29}$$

After the substitution $\beta=0$ (29) is reduced to the breather solution of the one-dimensional sine-Gordon equation like the solution (27). This fact bring about the conclusion that the connection between the classes of solutions A and B of the two-dimensional sine-Gordon equation exist.

2. $k_1[1-1/A^4]^{1/2}$; $k_2=1$. Then

$$\alpha^2=\left(\frac{A^2}{A^2+1}\right) \qquad \delta^2\gamma^2=\beta^2+\frac{A^2}{A^2+1}$$

and the solution ϕ_2 is reduced to the following solution:

$$\phi_{2,2}=4\tan^{-1}\left\{A\,\mathrm{dn}\left[\frac{A^2x}{A^2+1};\left(1-\frac{1}{A^4}\right)^{1/2}\right]\right.$$
$$\left.\times\tanh\left[\beta y\pm\left(\frac{A^2}{(A^2+1)^2}+\beta^2\right)^{1/2}t\right]\right\}. \tag{30}$$

$\phi_{2,2}$ is the two-dimensional generalization of the next solution of the one-dimensional sine-Gordon equation:

$$\bar{\phi}_{2,2}=4\tan^{-1}\left\{A\,\mathrm{dn}\left[\frac{A^2x}{A^2+1};\left(1-\frac{1}{A^4}\right)^{1/2}\right]\tanh\left[\pm\left(\frac{At}{A^2+1}\right)\right]\right\}. \tag{31}$$

The solution ϕ_4: there is a difference between the solutions ϕ_2 and ϕ_4. The solution ϕ_4 is not reduced to the breather solution (29). Only one special case is possible for

$$\phi_4\colon k_1=\left[1-\frac{1}{A^4}\right]^{1/2} \qquad k_2=0.$$

Then

$$\alpha^2=\frac{A^2}{A^2+1} \qquad \delta^2\gamma^2=\beta^2+\frac{A^2}{(A^2+1)^2}$$

and ϕ_4 is reduced to the following solution of equation (2):

$$\phi_{4,1}=4\tan^{-1}\left\{A\,\mathrm{dn}\left[\frac{A^2x}{A^2+1};\left(1-\frac{1}{A^4}\right)^{1/2}\right]\right.$$
$$\left.\times\coth\left[\beta y\pm\left(\frac{A^2}{(A^2+1)^2}+\beta^2\right)^{1/2}t\right]\right\}. \tag{32}$$

The solution (32) is a generalization of the next solution of equation (1):

$$\bar{\phi}_{4,1}=4\tan^{-1}\left\{A\,\mathrm{dn}\left[\frac{A^2x}{A^2+1};\left(1-\frac{1}{A^4}\right)^{1/2}\right]\coth\left[\pm\left(\frac{At}{A^2+1}\right)\right]\right\}. \tag{33}$$

Class C. The solution ϕ_3: two special cases are possible here:

1. $k_1=k_2=1$. Then

$$\alpha^2=\frac{A^2}{A^2-1} \qquad \delta^2\gamma^2=\beta^2+\frac{1}{A^2-1}$$

and the solution ϕ_3 is reduced to:

$$\phi_3=\phi_{3,1}=4\tan^{-1}\left\{A\frac{\sinh\left[\beta y\pm\left(\beta^2+\frac{1}{A^2-1}\right)^{1/2}t\right]}{\cosh\left[\left(\frac{A^2}{A^2-1}\right)^{1/2}x\right]}\right\}. \tag{34}$$

After the substitution $\beta=0$ into (34) the result is as follows:

$$\bar{\phi}_{3,1}=4\tan^{-1}\left\{A\frac{\sinh\left[\pm\left(\frac{1}{A^2-1}\right)^{1/2}t\right]}{\cosh\left[\left(\frac{A^2}{A^2-1}\right)^{1/2}x\right]}\right\}. \tag{35}$$

Equation (35) is a well known solution of equation (1). It describes soliton–antisoliton collisions (fluxon–antifluxon collisions in a one-dimensional Josephson junction). The solution (34) is a two-dimensional generalization of the soliton–antisoliton collision.

2. $k_1=(1-1/A^4)^{1/2}$; $k_2=0$. Then

$$\alpha^2=\frac{A^4}{(A^2-1)^2};\quad \delta^2\gamma^2=\beta^2+\frac{A^2}{(A^2-1)^2}$$

and the solution ϕ_3 is reduced to a solution consisting of a function of the product of a Jacobi elliptic function and an elementary function:

$$\phi_{3,2}=4\tan^{-1}\left\{A\,\mathrm{dn}\left[\frac{A^2x}{A^2-1}:\left(1-\frac{1}{A^4}\right)^{1/2}\right]\right.$$
$$\left.\times\tan\left[\beta y\pm\left(\beta^2+\frac{A^2}{(A^2-1)^2}\right)^{1/2}t\right]\right\}. \tag{36}$$

But now the elementary function is not hyperbolic. It is a trigonometric function. The solution (36) is a generalization of the corresponding solution of equation (1):

$$\bar{\phi}_{3,2}=4\tan^{-1}\left\{A\,\mathrm{dn}\left[\frac{A^2x}{A^2-1};\quad\left(1-\frac{1}{A^4}\right)^{1/2}\right]\tan\left[\pm\left(\frac{At}{A^2-1}\right)\right]\right\}. \tag{37}$$

The solution ϕ_5: an important difference between the solutions ϕ_3 and ϕ_5 is that the solution ϕ_5 is not reduced to the soliton–antisoliton collision. Only one special case is possible here:

$$k_1=\left(1-\frac{1}{A^4}\right)^{1/2};\quad k_2=0.$$

Then

$$\alpha^2=\frac{A^4}{(A^2-1)^2};\quad \delta^2\gamma^2=\beta^2+\frac{A^2}{(A^2-1)^2}$$

and ϕ_5 is reduced to the following solution of the two-dimensional sine-Gordon equation:

$$\phi_{5,1}=4\tan^{-1}\left\{A\,\mathrm{dn}\left[\frac{A^2x}{A^2-1};\quad\left(1-\frac{1}{A^4}\right)^{1/2}\right]\right.$$
$$\left.\times\cot\left[\beta y\pm\left(\beta^2+\frac{A^2}{(A^2-1)^2}\right)^{1/2}t\right]\right\}. \tag{38}$$

If $\beta=0$ the $\phi_{5,1}$ is reduced to the corresponding solution of the one-dimensional sine-Gordon equation:

$$\bar{\phi}_{5,1}=4\tan^{-1}\left\{A\,\mathrm{dn}\left[\frac{A^2x}{A^2-1};\quad\left(1-\frac{1}{A^4}\right)^{1/2}\right]\cot\left[\left(\pm\frac{At}{A^2-1}\right)\right]\right\}. \tag{39}$$

Class D. The solution ϕ_6: the following special case is possible here: $k_1 = k_2 = 1$. Then

$$\alpha^2 = \frac{1}{1-A}; \; \delta^2\gamma^2 = \beta^2 + \frac{A^2}{1-A^2}$$

and ϕ_6 is reduced to the solution:

$$\phi_{6,1} = 4\tan^{-1}\left\{A \frac{\sinh\left[\left(\frac{1}{1-A^2}\right)^{1/2} x\right]}{\cosh\left[\beta y \pm \left(\beta^2 + \frac{A^2}{1-A^2}\right)^{1/2} t\right]}\right\}. \tag{40}$$

If $\beta = 0$ the result is:

$$\bar{\phi}_{6,1} = 4\tan^{-1}\left\{A \frac{\sinh\left[\left(\frac{1}{1-A^2}\right)^{1/2} x\right]}{\cosh\left[\pm\left(\frac{A^2}{1-A^2}\right)^{1/2} t\right]}\right\}. \tag{41}$$

$\bar{\phi}_{6,1}$ is a solution of the one-dimensional sine-Gordon equation. It describes a soliton–soliton collision. Another name for this solution is a 4π-impulse. Hence $\phi_{6,1}$ is a two-dimensional generalization of the 4π-impulse solution.

The solution ϕ_7: an important difference between the solutions ϕ_6 and ϕ_u is that ϕ_7 is not reduced to a two-dimensional or one-dimensional 4π-impluse.

Using an approach for solving the two-dimensional sine-Gordon equation, seven solutions of this equation were found. These solutions describe running waves and they are generalizations of the solutions of the one-dimensional sine-Gordon equation. The solutions are distributed into four classes, analogous to the corresponding classes of solutions of equation (1). Two types of special case for the solutions of the two-dimensional sine-Gordon equation exist. The first type includes solutions of the equation (2) consisting only of elementary functions. These solutions are generalizations of the solutions of the one-dimensional sine-Gordon equation describing a breather, a soliton–antisolition collision and 4π-impulse. The second type of special case includes solutions of equation (2) consisting of a function of a product of an Jacobi elliptic function and an elementary function. These solutions are also generalizations of the corresponding solutions of the one-dimensional sine-Gordon equation.

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